RATIONALITY OF INSTANTON MODULI

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1. Introduction

By a *mathematical instanton of charge n*, we understand a rank-2 algebraic vector bundle E on the 3-dimensional projective space $\mathbb{P}^3$ with Chern classes

$$c_1(E) = 0, \quad c_2(E) = n, \tag{1}$$

satisfying the vanishing conditions

$$h^0(E) = h^1(E(-2)) = 0. \tag{2}$$

Such objects exist only if $n \geq 1$. The adjective "mathematical" serves to distinguish them from physical instantons, which are the vector bundles E as above plus a kind of a real structure: $\sigma^*(E) \simeq \overline{E}$, where σ is an anti-holomorphic involution on $\mathbb{P}^3$, and E is trivial on the σ-stable projective lines. The physical instantons are exactly the objects arising from the anti-self-dual $SU(2)$-connections on S^4 via the Atiyah–Ward correspondence. Their moduli space can be thought of as an open subset of the real locus of the complex moduli space I_n of mathematical instantons.

Throughout the paper, we will omit the epithet "mathematical" when speaking about mathematical instantons; we will also say n-instantons to specify the value of c_2, equal to n. The definition of an instanton implies that any instanton E is stable in the sense of Gieseker-Maruyama. Hence I_n is a subset of the moduli scheme $M_{\mathbb{P}^3}(2; 0, 2, 0)$ of semistable rank-2 torsion-free sheaves on $\mathbb{P}^3$ with Chern classes $c_1 = 0,\ c_2 = n,\ c_3 = 0$. The condition $h^1(E(-2)) = 0$ for $[E] \in I_n$ (called the *instanton condition*) by the semicontinuity implies that I_n is a Zariski open subset of $M_{\mathbb{P}^3}(2; 0, 2, 0)$, i.e. I_n is a quasiprojective scheme. The expected dimension of I_n is $8n - 3$. It is known that the moduli space of physical instantons is irreducible and that $\mathrm{Ext}^2(E, E) = 0$ when E is a physical instanton ([ADHM], [DM]; see also Ch. II, Sect. 4.4.3 in [OSS]). Thus I_n has at least one component of expected dimension, the one containing the physical instantons. It is conjectured that I_n is irreducible and rational for all $n \geq 1$.

The irreducibility of I_n is quite a resistant problem. As of today, it has been proved for $n \leq 5$ and for all odd n, see [B2] (n=1), [H] (n=2), [ES] (n=3), [B3] (n=4), [CTT] (n=5), [T] (odd $n \geq 1$). For $1 \leq n \leq 3$, the same papers provide the rationality of I_n. The rationality of I_5 was proved under the irreducibility hypothesis by Katsylo in [Ka]. In the present paper, we prove the following result:

Theorem 1.1. *Whenever I_n is irreducible, it is rational. In particular, I_n is rational for all odd $n \geq 1$ and for $n = 2, 4$.*

In Section 2, we explain the two main reduction steps in our proof of rationality: (G, H)-slice and Noname Lemma. The first allows us to represent, from the birational point of view, the quotient by an algebraic group G as the quotient over a smaller group H. The second is a descent technique for a fibration structure under the quotient map, in the case when the fibration is an affine bundle. Though this tool should be well known to specalists, we found only a proof for the case of vector bundles in the literature, so we provide a detailed proof of the version for the affine bundles. In Section 3, we introduce Barth's nets of quadrics, which are the basic linear algebraic data for obtaining instantons as the cohomology of a skew-symmetric monad. In Section 4, we represent I_n as a quotient of some set of linear-algebraic data by

a free action of the group $G = GL_n \times Sp_{2n+2}/\{\pm(1,1)\}$. In Section 5, we describe Barth's construction of a (G,H)-slice which allows us to go over to a quotient by a smaller group $H \subset G$, $H \simeq O_n \times Sp_2/\{\pm(1,1)\}$. Finally, in Section 6, we introduce on our (G,H)-slice a structure of an affine bundle and apply the Noname Lemma. The latter provides a descent of the affine bundle to an affine bundle over an open subset of the moduli space of vector bundles on $\mathbb{P}^2$, which is known to be rational. This observation permits to end the proof of Theorem 1.1.

Notation and conventions. Our notation is mostly standard. The base field $\mathbf{k}$ is assumed to be algebraically closed of characteristic 0. A variety is a reduced scheme of finite type over $\mathbf{k}$. An algebraic group is a variety with morphisms of the group law and the inverse, satisfying standard axioms. We identify vector bundles with locally free sheaves. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules on an algebraic variety or scheme X, then $\mathcal{F}^\vee$ denotes the dual to $\mathcal{F}$ sheaf, i.e. the sheaf $\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. If $X = \mathbb{P}^r$ and t is an integer, then by $\mathcal{F}(t)$ we denote the sheaf $\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^r}(t)$. $[\mathcal{F}]$ will denote the isomorphism class of a sheaf $\mathcal{F}$. For any morphism of $\mathcal{O}_X$-sheaves $f : \mathcal{F} \to \mathcal{F}'$ and any $\mathbf{k}$-vector space U (respectively, for any homomorphism $f : U \to U'$ of $\mathbf{k}$-vector spaces) we will denote, for short, the induced morphism of sheaves $id \otimes f : U \otimes \mathcal{F} \to U \otimes \mathcal{F}'$ (respectively, the induced morphism $f \otimes id : U \otimes \mathcal{F} \to U' \otimes \mathcal{F}$) by the same symbol f.

Everywhere in the paper V will denote a fixed vector space of dimension 4 over $\mathbf{k}$, and we set $\mathbb{P}^3 := P(V)$. Also everywhere below we will reserve the letters u and v for denoting the two morphisms in the Euler exact sequence $0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{u} V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{v} T_{\mathbb{P}^3}(-1) \to 0$.

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2. Two reduction tools for proving rationality

We recall that, by definition, an algebraic variety X over $\mathbf{k}$ is called rational if it is irreducible and has a Zariski open subset, isomorphic to a Zariski open subset of the affine space $\mathbb{A}^n_{\mathbf{k}}$, where $n = \dim X$. Equivalently, the algebra of rational functions of X is isomorphic to $\mathbf{k}(T_1, \dots, T_n)$, a purely transcendental extension of $\mathbf{k}$.

The instanton moduli space will be described as a quotient of some locally closed subvariety of the affine space by an action of an algebraic group G. Dolgachev's paper [D] provides a survey of various methods for proving the rationality of quotient spaces. We will review two of them, the ones used in our paper.

2.1. (G,H)-slices

This method was used by Bogomolov and Katsylo in different problems of the geometric invariant theory, in particular, in the proof of the rationality of I_5, see [BK], [Ka]. Later we will use a (G,H)-slice of a quotient representation of I_n for arbitrary n, constructed by Barth in [B3].

Let X be an irreducible algebraic variety, $Y \subset X$ a closed irreducible subvariety, G an algebraic group acting on X, and H a closed subgroup of G.

Definition 2.1. Y is called a (G,H)-slice of the action $G : X \circlearrowleft$ if the following two conditions are verified:

(i) $\overline{G \cdot Y} = X$, and

(ii) there exists an open subset $Y_0 \subset Y$ such that for $y \in Y_0$, we have
$$g \in G,\ gy \in Y \quad \Longleftrightarrow \quad g \in H.$$

Proposition 2.2. *If Y is a (G, H)-slice of the action $G : X \circlearrowleft$, then $\mathbf{k}(X)^G \simeq \mathbf{k}(Y)^H$.*

Proof. See [GV], Proposition 4. □

2.2. Noname Lemma

The term belongs to Dolgachev [D], but the method has been extensively used by many authors. It is also called "descent for vector bundles" or Kempf Lemma ([DN], Th. 2.3), but variants of it for G a Galois group were used earlier in [L] and [Sp]. Dolgachev states it for a G-linearized vector bundle $X \to V$; we use a generalization for an affine bundle. So we provide the statement and a proof of this generalized version, though the proof is essentially the same as in the case of a vector bundle. We first fix some terminology and recall some generalities on principal bundles in the framework of algebraic geometry, where they are usually called torsors.

Let E be a scheme over $\mathbf{k}$ and H an algebraic group acting on E. The action is free if the map $\Psi = (\mu, \mathrm{id}_E) : H \times E \to E \times E$ is a closed immersion, where $\mu : H \times E \to E$ is the action map (we write the definitions for a left action). We recall also that $E \to V$ is called a (left) H-torsor if a (left) action $\mu : H \times E \to E$ of H on E is given such that $E \to V$ is flat and surjective and the above map Ψ is a closed immersion inducing an isomorphism $H \times E \simeq E \times_V E$.

The definition of a torsor immediately implies that any torsor $E \to V$ is locally trivial in the fpqc and in fppf topologies. Indeed, $E \to V$ is a fpqc and a fppf morphism, so it can be considered as a fpqc- (or a fppf-) covering of V consisting of a single chart, and the base change of $E \to V$ by itself trivializes it.

An algebraic group H is called special in the sense of Grothendieck and Serre if any H-torsor is locally trivial in the Zariski topology. Grothendieck provides a homological characterization of special groups in [Gr]. From this characterization, it follows that: 1) any special group is connected; 2) H is special if and only if H/R_u is special, where R_u denotes the unipotent radical of H; 3) a direct product of special groups is special; 4) the groups GL_n, SL_n, Sp_n are special.

Let H_r denote the group Aff $(\mathbb{A}^r_{\mathbf{k}})$ of affine transformations of the affine space $\mathbb{A}^r_{\mathbf{k}}$ of dimension r. A morphism $E \to V$ is called an affine bundle of rank r if there exists an H_r-torsor $E \to V$ such that $X \to V$ is the fiber bundle with fiber $\mathbb{A}^r_{\mathbf{k}}$, associated to $E \to V$ via the natural action of H_r on $\mathbb{A}^r_{\mathbf{k}}$.

As $H_r/R_u(H_r) \simeq GL_r$, it follows from 2) that H_r is special. The speciality of GL_n is equivalent to the well known fact that any vector bundle is locally trivial in Zariski topology. The speciality of H_r implies the local triviality of any affine bundle.

Lemma 2.3 (Noname Lemma)**.** *Let V be an irreducible algebraic variety over $\mathbf{k}$, $\phi : X_V \to V$ an affine bundle of rank r, G an algebraic group acting on X_V, V in such a way that ϕ is G-equivariant and the fibers of $X_V \to V$ are acted on by affine isomorphisms. Assume that the stabilizer of a generic point of V in G is trivial. Then there exists a G-stable open subset U in V such that: (a) the action of G on U is free, so that there is a quotient map $U \to U/G$, which is a G-torsor; (b) the restricted affine bundle $\phi^{-1}(U) \to U$ is G-equivariantly trivial, that is, there exists an isomorphism of affine bundles $\phi^{-1}(U) \simeq \mathbb{A}^r_{\mathbf{k}} \times U$ which is at the same time an isomorphism of G-varieties, the action of G on $\mathbb{A}^r_{\mathbf{k}}$ being trivial.*

In particular, $\phi^{-1}(U)/G \simeq \mathbb{A}^r_{\mathbf{k}} \times (U/G)$, and $\mathbf{k}(X_V)^G \simeq \mathbf{k}(V)^G(T_1, \ldots, T_r)$ is a purely transcendental extension of $\mathbf{k}(V)^G$.

Proof. By [R], there is a G-stable Zariski open subset V_0 of V such that the geometric quotient $Y_0 = V_0/G$ exists. This means, by definition, that: 1) there is a surjective morphism $\pi : V_0 \to Y_0$ and the map $\Psi : G \times V_0 \to V_0 \times_{Y_0} V_0$ defined as above is surjective; 2) π is submersive; 3)

$\mathcal{O}_{Y_0} = \pi_* \mathcal{O}_{V_0}^G$. In particular, the fibers of π are exactly G-orbits in V_0. By the hypothesis on the stabilizers, we may shrink further Y_0 so that Ψ is also injective. As Ψ is now bijective, it restricts to an isomorphism of some open subsets. The locus where Ψ fails to be a local isomorphism is a union of G-orbits, because G acts by isomorphisms on V_0. Hence we can further shrink Y_0 so that Ψ is an isomorphism, and Y_0, π are smooth. Then, a fortiori, π is flat and, by definition, is a torsor. Replacing V by $V_0 = \pi^{-1}(Y_0)$, we may assume that there is a quotient map $\pi : V \to Y = V/G$ which is a G-torsor.

Now the action of G on X_V lifting the action on V can be thought of as a descent datum for a scheme over Y, defined on the fpqc covering $\{V \xrightarrow{\pi} Y\}$ of Y (a covering consisting of a single fpqc chart). Indeed, a descent datum (see [SGA1], Ch. VIII, or Appendix A in [Stacks]) for an object X over Y is the following triple: a fpqc covering $Y' \to Y$, an object $X' \to Y'$ over Y', and a transition isomorphism $f : X' \times_Y Y' \to Y' \times_Y X'$ satisfying the cocycle condition $f_{13} = f_{23} f_{12}$, where $f_{12} : X' \times_Y Y' \times_Y Y' \to Y' \times_Y X' \times_Y Y'$ is $f \times_Y Y'$, and the other maps f_{ij} are defined similarly. In our situation, $Y' = V$ $X' = X_V$ and $f : X_V \times_Y V \to V \times_Y X_V$ is defined on points by $(x, v) \mapsto (\phi(x), gx)$, where $g = g(x, v)$ is the unique element of G such that $v = g\phi(x)$.

By loc. sit., every descent datum for an affine scheme over Y is effective, that is realized by an affine scheme X over Y together with an isomorphism $\lambda : X \times_Y Y' \to X'$ such that $Y' \times_Y \lambda = f \circ (\lambda \times_Y Y')$. We obtain the existence of the affine morphism $X/G \to V/G = Y$, the quotient of $X_V \to V$.

The fact that the structure of the affine bundle descends to the quotient is obtained in a similar way. As H is special, the affine bundle $X \to V$ is trivialized locally on a covering $V' \to V$ by Zariski open subsets. The trivialization is an isomorphism $\lambda_{V'} : X \times_V V' \simeq \mathbb{A}^r_{\mathbf{k}} \times V' =: \mathbb{A}_{V'}$ compatible with the transition map $\psi : \mathbb{A}_{V'} \times_V V' \to V' \times_V \mathbb{A}_{V'}$ in the following sense: $V' \times_V \lambda_{V'} = \psi \circ (\lambda_{V'} \times_V V')$. Moreover, ψ is the left action by a function $s : V' \times_V V' \to H = H_r$ satisfying the cocycle condition $\mathrm{pr}_{23}^*(s)\mathrm{pr}_{12}^*(s) = \mathrm{pr}_{13}^*(s)$ on $V' \times_V V' \times_V V'$.

The action of G on X preserving the structure of an affine bundle can be defined by adding to the covering the open sets gU, where U runs over all the open sets of the covering $V' \to V$ and g over all of G. We will denote this extended covering be the same symbol.

Again, by the *existence* of a descent for affine schemes, $V' \to V$ and $\lambda_{V'}$ descend to $Y' \to Y$ and $\lambda_{Y'}$. All the properties expressing the fact that $\lambda_{V'}$ is an affine trivialization descend by the *uniqueness* of a descent for morphisms of schemes, for they are expressed in terms of equalities between some morphisms.

Finally, we have proved that $X/G \to Y$ is an affine bundle. Using again the speciality of H, we conclude that there exists a Zariski open subset of Y of the form U/G with wanted properties.

$\square$

3. Instantons and nets of quadrics

Now we will recall some well known facts about instantons, see, e.g., [CTT] and further references therein.

For a given n-instanton E, the conditions (1), (2), Riemann-Roch and Serre duality imply

$$h^1(E(-1)) = h^2(E(-3)) = n, \quad h^1(E \otimes \Omega^1_{\mathbb{P}^3}) = h^2(E \otimes \Omega^2_{\mathbb{P}^3}) = 2n + 2, \qquad (3)$$
$$h^1(E) = h^2(E(-4)) = 2n - 2.$$

Furthermore, the condition $c_1(E) = 0$ yields an isomorphism $\wedge^2 E \xrightarrow{\simeq} \mathcal{O}_{\mathbb{P}^3}$, hence a symplectic isomorphism $j : E \xrightarrow{\simeq} E^\vee$. This symplectic structure j on E is unique up to a scalar, since E as a stable bundle is a simple bundle, i.e. $\mathrm{Hom}(E, E) = \mathbf{k}id$. Consider a triple (E, f, j)

where E is an n-instanton, f is an isomorphism $H_n \stackrel{\simeq}{\to} H^2(E(-3))$ and $j : E \stackrel{\simeq}{\to} E^\vee$ is a symplectic structure on E. We call two such triples (E, f, j) and $(E' f', j')$ equivalent if there is an isomorphism $g : E \stackrel{\simeq}{\to} E'$ such that $g_* \circ f = \lambda f'$ with $\lambda \in \{1, -1\}$ and $j = g^\vee \circ j' \circ g$, where $g_* : H^2(E(-3)) \stackrel{\simeq}{\to} H^2(E'(-3))$ is the induced isomorphism. We denote by $[E, f, j]$ the equivalence class of a triple (E, f, j). From this definition one easily deduces that the set $F_{[E]}$ of all equivalence classes $[E, f, j]$ with given $[E]$ is a homogeneous space of the group $GL(H_n)/\{\pm id\}$.

Each class $[E, f, j]$ defines a point

$$A_n = A_n([E, f, j]) \in S^2 H_n^\vee \otimes \wedge^2 V^\vee \tag{4}$$

in the following way. Consider the exact sequences

$$0 \to \Omega^1_{\mathbb{P}^3} \stackrel{i_1}{\to} V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3} \to 0, \tag{5}$$

$$0 \to \Omega^2_{\mathbb{P}^3} \to \wedge^2 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-2) \to \Omega^1_{\mathbb{P}^3} \to 0, 0 \to \wedge^4 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-4) \to \wedge^3 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-3) \stackrel{i_2}{\to} \Omega^2_{\mathbb{P}^3} \to 0,$$

induced by the Koszul complex of $V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \stackrel{ev}{\twoheadrightarrow} \mathcal{O}_{\mathbb{P}^3}$. Twisting these sequences by E and passing to cohomoligy in view of (2) gives the diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & H^2(E(-4)) \otimes \wedge^4 V^\vee \arrow[r] & H^2(E(-3)) \otimes \wedge^3 V^\vee \arrow[r, "i_2"] \arrow[d, "A'"] & H^2(E \otimes \Omega^2_{\mathbb{P}^3}) \arrow[r] & 0 \\ 0 & H^1(E)) \arrow[l] & H^1(E(-1)) \otimes V^\vee \arrow[l] & H^1(E \otimes \Omega_{\mathbb{P}^3}) \arrow[l, "i_1"] \arrow[u, "\cong", "\partial"'] & 0, \arrow[l] \end{tikzcd} \tag{6}$$

where $A' := i_1 \circ \partial^{-1} \circ i_2$. The Euler exact sequence (5) yields the canonical isomorphism $\omega_{\mathbb{P}^3} \stackrel{\simeq}{\to} \wedge^4 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-4)$, and fixing an isomorphism $\tau : \mathbf{k} \stackrel{\simeq}{\to} \wedge^4 V^\vee$ induces the isomorphisms $\tilde{\tau} : V \stackrel{\simeq}{\to} \wedge^3 V^\vee$ and $\hat{\tau} : \omega_{\mathbb{P}^3} \stackrel{\simeq}{\to} \mathcal{O}_{\mathbb{P}^3}(-4)$. Now the point $A = A_n$ in (4) is defined as the composition

$$A : H_n \otimes V \overset{\tilde{\tau}}{\underset{\simeq}{\to}} H_n \otimes \wedge^3 V^\vee \overset{f}{\underset{\simeq}{\to}} H^2(E(-3)) \otimes \wedge^3 V^\vee \stackrel{A'}{\to} H^1(E(-1)) \otimes V^\vee \overset{j}{\underset{\simeq}{\to}} \tag{7}$$

$$\overset{j}{\underset{\simeq}{\to}} H^1(E^\vee(-1)) \otimes V^\vee \overset{SD}{\underset{\simeq}{\to}} H^2(E(1) \otimes \omega_{\mathbb{P}^3})^\vee \otimes V^\vee \overset{\hat{\tau}}{\underset{\simeq}{\to}} H^2(E(-3))^\vee \otimes V^\vee \overset{f^\vee}{\underset{\simeq}{\to}} H_n^\vee \otimes V^\vee,$$

where SD is the Serre duality isomorphism. One checks that A_n is a skew symmetric map depending only on the class $[E, f, j]$ and not depending on the choice of τ, and that this point $A_n \in \wedge^2(H_n^\vee \otimes V^\vee)$ lies in the direct summand

$$\mathbf{S}_n := S^2 H_n^\vee \otimes \wedge^2 V^\vee$$

of the canonical decomposition

$$\wedge^2 (H_n^\vee \otimes V^\vee) = S^2 H_n^\vee \otimes \wedge^2 V^\vee \oplus \wedge^2 H_n^\vee \otimes S^2 V^\vee. \tag{8}$$

Here $\mathbf{S}_n$ is the space of nets of quadrics in H_n. Following [B3], [Tju2] and [Tju1] we call A the *n-instanton net of quadrics* corresponding to the data $[E, f, j]$.

Denote $W_A := H_n \otimes V / \ker A$. Using the above chain of isomorphisms we can rewrite the diagram (6) as

$$\begin{tikzcd} 0 \arrow[r] & \ker A \arrow[r] & H_n \otimes V \arrow[r, "c_A"] \arrow[d, "A"] & W_A \arrow[r] \arrow[d, "\cong", "q_A"'] & 0 \\ 0 & \ker A^\vee \arrow[l] & H_n^\vee \otimes V^\vee \arrow[l] & W_A^\vee \arrow[l, "c_A^\vee"'] & 0. \arrow[l] \end{tikzcd} \tag{9}$$

Here $\dim W_A = 2n+2$ and $q_A : W_A \overset{\simeq}{\to} W_A^\vee$ is the induced skew-symmetric isomorphism. An important property of $A = A_n([E,f,j])$ is that the induced morphism of sheaves

$$a_A^\vee : W_A^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \overset{c_A^\vee}{\to} H_n^\vee \otimes V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \overset{ev}{\to} H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \tag{10}$$

is an epimorphism such that the composition $H_n \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \overset{a_A}{\to} W_A \otimes \mathcal{O}_{\mathbb{P}^3} \overset{q_A}{\to} W_A^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \overset{a_A^\vee}{\to} H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1)$ is zero, and $E = \ker(a_A^\vee \circ q_A)/\operatorname{Im} a_A$. Thus A defines a monad

$$\mathcal{M}_A : \quad 0 \to H_n \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \overset{a_A}{\to} W_A \otimes \mathcal{O}_{\mathbb{P}^3} \overset{a_A^\vee \circ q_A}{\to} H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0 \tag{11}$$

with the cohomology sheaf E,

$$E = E(A) := \ker(a_A^\vee \circ q_A)/\operatorname{Im} a_A. \tag{12}$$

Note that passing to cohomology in the monad $\mathcal{M}_A$ twisted by $\mathcal{O}_{\mathbb{P}^3}(-3)$ and using (12) yields the isomorphism $f : H_n \overset{\simeq}{\to} H^2(E(-3))$. Furthermore, the nondegeneracy of q_A in the monad $\mathcal{M}_A$ implies that there is a canonical isomorphism of $\mathcal{M}_A$ with its dual which induces the symplectic isomorphism $j : E \overset{\simeq}{\to} E^\vee$. Thus, the data $[E,f,j]$ are recovered from the net A. This leads to the following description of the moduli space I_n. Consider the *set of n-instanton nets of quadrics*

$$MI_n := \left\{ A \in \mathbf{S}_n \;\middle|\; \begin{array}{c} (i)\ \mathrm{rk}(A : H_n \otimes V \to H_n^\vee \otimes V^\vee) = 2n+2, \\ (ii)\ \text{the morphism } a_A^\vee : W_A^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \to H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \\ \text{defined by } A \text{ in (10) is surjective,} \\ (iii)\ h^0(E_2(A)) = 0, \text{ where } E_2(A) := \ker(a_A^\vee \circ q_A)/\operatorname{Im} a_A \\ \text{and } q_A : W_A \overset{\simeq}{\to} W_A^\vee \text{ is a symplectic isomorphism} \\ \text{defined by } A \text{ in (9)} \end{array} \right\} \tag{13}$$

The conditions (i)-(iii) here are called *Barth's coditions.* These conditions show that MI_n is naturally supplied with a structure of a locally closed subscheme of the vector space $\mathbf{S}_n$. Moreover, the above description shows that there is a morphism $\pi_n : MI_n \to I_n : \ A \mapsto [E(A)]$, and it is known that this morphism is a principal $GL(H_n)/\{\pm id\}$-bundle in the étale topology - cf. [CTT]. Here by construction the fibre $\pi_n^{-1}([E])$ over an arbitrary point $[E] \in I_n$ is a principal homogeneous space $F_{[E]}$ of the group $GL(H_n)/\{\pm id\}$ described above.

The definition (13) yields the following.

Theorem 3.1. *For each $n \geq 1$, the space of n-instanton nets of quadrics MI_n is a locally closed subscheme of the vector space $\mathbf{S}_n$ given locally at any point $A_n \in MI_n$ by*

$$\binom{2n-2}{2} = 2n^2 - 5n + 3 \tag{14}$$

equations obtained as the rank condition (i) in (13).

Note that from (14) it follows that

$$\dim_{[A]} MI_n \geq \dim \mathbf{S}_n - (2n^2 - 5n + 3) = n^2 + 8n - 3 \tag{15}$$

at any point $A_n \in MI_n$. On the other hand, by deformation theory for any n-instanton E we have $\dim_{[E]} I_n \geq 8n-3$. This agrees with (15), since $MI_n \to I_n$ is a principal $GL(H_n)/\{\pm id\}$-bundle in the étale topology.

4. Extending the structure group by Sp_{2n+2}

The previous linear algebra datum A for the description of instantons permitted to recover the $(2n+2)$-dimensional symplectic vector space (W_A, q_A), but the latter came without any pre-selected basis. In this section, we will provide another linear algebraic datum encoding the monad 11 together with a fixed choice of a basis in $W_A = W_{2n+2}$ and a fixed $q = q_A$.

Fix a $(2n+2)$-dimensional vector space W_{2n+2} and a symplectic structure q on W_{2n+2}, i.e. a skew-symmetric isomorphism $q : W_{2n+2} \xrightarrow{\cong} W_{2n+2}^\vee$. Consider the vector space $U := H_n^\vee \otimes W_{2n+2} \otimes V^\vee$ and its open subset $U_0 := \{\gamma \in U |\ \gamma$ understood as a homomorphism $H_n \otimes V \to W_{2n+2}$ is surjective$\}$. For an arbitrary point $\gamma \in U_0$ consider the composition

$$A(\gamma) : H_n \otimes V \overset{\gamma}{\twoheadrightarrow} W_{2n+2} \underset{\sim}{\xrightarrow{q}} W_{2n+2}^\vee \overset{\gamma^\vee}{\hookrightarrow} H_n^\vee \otimes V^\vee, \quad A(\gamma) \in \wedge^2(H_n^\vee \otimes V^\vee). \tag{16}$$

Since the symplectic form q on W_{2n+2} is an isomorphism, the condition $\gamma \in U_0$ is clearly equivalent to the condition

$$\mathrm{rk} A(\gamma) = \dim W_{2n+2} = 2n+2. \tag{17}$$

Besides, consider the composition

$$\alpha(\gamma) : H_n \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{u} H_n \otimes V \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\gamma} W_{2n+2} \otimes \mathcal{O}_{\mathbb{P}^3}. \tag{18}$$

Define the space

(19)

$$MI_n^{sp} := \left\{ \gamma \in U \;\middle|\; \begin{array}{l} (i)\ \mathrm{rk} A(\gamma) = 2n+2, \\ (ii)\ \alpha(\gamma)^\vee \circ q \circ \alpha(\gamma) = 0, \\ (iii)\ \text{the morphism } \alpha(\gamma)^\vee : W^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \to H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \text{ is surjective}, \\ (iv)\ h^0(E(\gamma)) = 0, \text{ where } E(\gamma) := \ker(\alpha(\gamma)_A^\vee \circ q)/\operatorname{Im}\alpha(\gamma) \end{array} \right\}$$

Note that the conditions (ii)-(iv) in the definition of MI_n^{sp} mean that, for any $\gamma \in MI_n^{sp}$, the sequence

$$0 \to H_n \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha(\gamma)} W_{2n+2} \otimes \mathcal{O}_{\mathbb{P}^3} \underset{\sim}{\xrightarrow{q}} W_{2n+2}^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\alpha(\gamma)^\vee} H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0 \tag{20}$$

is a monad (of the same form as (11)) with the cohomology sheaf

$$E = E(\gamma) = \ker(\alpha(\gamma)_A^\vee \circ q)/\operatorname{Im}\alpha(\gamma), \quad [E] \in I_n. \tag{21}$$

Next, the group $\widetilde{G} := GL(H_n) \times Sp(W_{2n+2}, q)$ naturally acts on the vector space U, hence on MI_n^{sp}, and it is well known (see, e.g., [BH, p. 341, Remark 2]) that the stabilizer in $\widetilde{G}$ of any point $\gamma \in MI_n^{sp}$ coincides with the subgroup $\{(id_{H_n}, id_{W_{2n+2}}), (-id_{H_n}, -id_{W_{2n+2}})\} \simeq \mathbb{Z}_2$. Hence the group $G := \widetilde{G}/\mathbb{Z}_2$ acts freely on MI_n^{sp}, and in accordance with (21) we obtain the natural isomorphism $MI_n^{sp}/G \xrightarrow{\sim} I_n$ such that the projection

$$\lambda_n : \ MI_n^{sp} \to I_n : \ \gamma \mapsto [E(\gamma)] \tag{22}$$

is a principal G-bundle in the étale topology.

Let $pr_2 : \wedge^2(H_n^\vee \otimes V^\vee) \to \wedge^2 H_n^\vee \otimes S^2 V^\vee$ be the projection onto the second direct summand of the decomposition (8). Comparing (16) and (18) we see that the condition (ii) in (19) can be rewritten as

$$pr_2(A(\gamma)) = 0, \tag{23}$$

i.e., equivalently, as

$$A(\gamma) \in \mathbf{S}_n. \tag{24}$$

Whence, comparing the conditions (i)-(iv) in (19) with the conditions (i)-(iii) in (13) we obtain a well-defined morphism

$$\rho_n : \ MI_n^{sp} \to MI_n : \ \gamma \mapsto A(\gamma). \tag{25}$$

Note that the natural monomorphism of groups $Sp(W_{2n+2},q) \hookrightarrow \widetilde{G}$ gives an $Sp(W_{2n+2},q)$-action on MI_n^{sp} and map ρ_n is clearly $Sp(W_{2n+2},q)$-invariant. Show that, in fact, $MI_n \simeq MI_n^{sp}/Sp(W_{2n+2},q)$ and $\rho_n : MI_n^{sp} \to MI_n$ is a principal $Sp(W_{2n+2},q)$-bundle in the Zariski topology. Indeed, for a given point $A \in MI_n$ pick an isomorphism $\psi : W_{2n+2} \stackrel{\simeq}{\to} W_A$ making commutative the diagram

$$\begin{tikzcd} W_{2n+2} \arrow[r, "q", "\simeq"'] \arrow[d, "\psi"', "\simeq"] & W_{2n+2}^\vee \\ W_A \arrow[r, "q_A", "\simeq"'] & W_A^\vee \arrow[u, "\psi^\vee", "\simeq"'] \end{tikzcd} \tag{26}$$

Now the isomorphism ψ is defined uniquely up to a $Sp(W_{2n+2},q)$-action on W_{2n+2}. This means that the monad (11) defined by the point A lifts to a monad (20), i.e. to a point $\gamma \in \rho_n^{-1}(A)$ uniquely up to a $Sp(W_{2n+2},q)$-action. Hence $\rho_n^{-1}(A)$ is a principal homogeneous $Sp(W_{2n+2},q)$-space, and we get the above statement. Thus, denoting by χ_n the map $MI_n^{sp} \to I_n : \gamma \mapsto [E(\gamma)]$, we obtain the diagram of principal bundles in the étale topology

$$\begin{tikzcd}[column sep=huge] MI_n^{sp} \arrow[r, "Sp(W_{2n+2},q)-\text{bundle}", "\rho_n"'] \arrow[rr, bend right, "G-\text{bundle}", "\chi_n"'] & MI_n \arrow[r, "GL(H_n)/\{\pm 1\}-\text{bundle}", "\pi_n"'] & I_n \end{tikzcd} . \tag{27}$$

5. Barth's (G,H)-slice

The construction we describe below represents an open subset of I_n as the base of a principal bundle with a smaller structure group $H \subset G$. It was developed by Barth in [B3]. Take an arbitrary point

$$\mathcal{A} = (A_1, A_2, B_1, B_2, a_1, a_2, b_1, b_2) \in (S^2 H_n^\vee)^{\times 4} \times (H_n^\vee)^{\times 4}, \tag{28}$$

and fix a basis in H_n, so that A_i's and B_i's can be understood as symmetric $n \times n$-matrices and a_i's and b_i's as column n-vectors. We may consider the point $\mathcal{A}$ as a $(2n+2) \times 4n$-matrix of the following block form

$$\widetilde{\mathcal{A}} = \begin{pmatrix} \mathbf{0} & \mathbf{1}_n & A_1 & A_2 \\ -\mathbf{1}_n & \mathbf{0} & B_1 & B_2 \\ & & a_1^T & a_2^T \\ & & b_1^T & b_2^T \end{pmatrix}. \tag{29}$$

Next, take the $(2n+2) \times (2n+2)$-matrix Q of the block form

$$Q = \begin{pmatrix} \mathbf{0} & \mathbf{1}_n & & \\ -\mathbf{1}_n & \mathbf{0} & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix} \tag{30}$$

and consider the skew $4n \times 4n$-matrix

$$A(\mathcal{A}) := \widetilde{\mathcal{A}}^T \cdot Q \cdot \widetilde{\mathcal{A}} \tag{31}$$

having the block form, with $n \times n$ matrices as blocks,

$$A(\mathcal{A}) = \begin{pmatrix} \mathbf{0} & \mathbf{1}_n & A_1 & A_2 \\ -\mathbf{1}_n & \mathbf{0} & B_1 & B_2 \\ -A_1 & -B_1 & [A_1,B_1] + a_1 \wedge b_1 & C \\ -A_2 & -B_2 & -C^T & [A_2,B_2] + a_2 \wedge b_2 \end{pmatrix}, \tag{32}$$

where

$$C = C(\mathcal{A}) := A_1B_2 - B_1A_2 + a_1b_2^T - b_1a_2^T \tag{33}$$

and where we use the notation $a \wedge b := a \cdot b^T - b \cdot a^T$ for arbitrary column vectors $a, b \in H_n$

Now consider the set

$$\Gamma_n := \{\mathcal{A} = (A_1, A_2, B_1, B_2, a_1, a_2, b_1, b_2) \in (S^2H_n^\vee)^{\times 4} \times (H_n^\vee)^{\times 4} \mid \mathcal{A} \text{ satisfies } (i)_\Gamma - (iv)_\Gamma\}, \tag{34}$$

where

$$\begin{aligned} (i)_\Gamma: &\quad [A_1 + tA_2, B_1 + tB_2] + (a_1 + ta_2) \wedge (b_1 + tb_2) = 0,\ t \in \mathbf{k}, \\ (ii)_\Gamma: &\quad a_{A(\mathcal{A})} : H_n \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to W_{A(\mathcal{A})} \otimes \mathcal{O}_{\mathbb{P}^3} \text{ is a subbundle morphism}, \\ (iii)_\Gamma: &\quad h^0(E(\mathcal{A})) = 0, \\ (iv)_\Gamma: &\quad \mathrm{rk}(a_1 \wedge a_2) = \mathrm{rk}(b_1 \wedge b_2) = 2. \end{aligned} \tag{35}$$

(here we set $a \wedge b := a \cdot b^T - b \cdot a^T$ for arbitrary column vectors $a, b \in H_n$).

The condition $(i)_\Gamma$ can be rewritten as

$$[A_1, B_1] + a_1 \wedge b_1 = 0, \quad [A_2, B_2] + a_2 \wedge b_2 = 0, \tag{36}$$

$$[A_1, B_2] + [A_2, B_1] + a_1 \wedge b_2 + a_2 \wedge b_1 = 0. \tag{37}$$

The last equation (37) in view of (33) can be rewritten as

$$C = C^T. \tag{38}$$

Now let

$$\mathcal{A} \in \Gamma_n. \tag{39}$$

Then in view of (36) and (38) the matrix $A(\mathcal{A})$ in (32) can be rewritten as

$$A(\mathcal{A}) = \begin{pmatrix} \mathbf{0} & \mathbf{1}_n & A_1 & A_2 \\ -\mathbf{1}_n & \mathbf{0} & B_1 & B_2 \\ -A_1 & -B_1 & \mathbf{0} & C \\ -A_2 & -B_2 & -C & \mathbf{0} \end{pmatrix}. \tag{40}$$

Next, since a basis in H_n is fixed, we can understand the symmetric matrices A_1, A_2, B_1, B_2, C as selfdual homomorphisms $H_n \to H_n^\vee$. Hence, fixing also a basis e_1, e_2, e_3, e_4 we can understand the matrix $A(\mathcal{A})$ as the selfdual w.r.t. H_n homomorphism $A(\mathcal{A}) : H_n \otimes \wedge^2 V \to H_n^\vee$ defined by the equalities

$$\begin{aligned} &A(\mathcal{A})|_{H_n \otimes \mathbf{k}e_1 \wedge e_2} = \mathbf{1}_n, \quad A(\mathcal{A})|_{H_n \otimes \mathbf{k}e_1 \wedge e_3} = A_1, \quad A(\mathcal{A})|_{H_n \otimes \mathbf{k}e_1 \wedge e_4} = A_2, \\ &A(\mathcal{A})|_{H_n \otimes \mathbf{k}e_2 \wedge e_3} = B_1, \quad A(\mathcal{A})|_{H_n \otimes \mathbf{k}e_2 \wedge e_4} = B_2, \quad A(\mathcal{A})|_{H_n \otimes \mathbf{k}e_3 \wedge e_4} = C(\mathcal{A}). \end{aligned} \tag{41}$$

i.e. as the point

$$A(\mathcal{A}) \in \mathbf{S}_n. \tag{42}$$

Also, treating $A(\mathcal{A})$ as a skew symmetric homomorphism $A(\mathcal{A}) : H_n \otimes V \to H_n^\vee \otimes V^\vee$ and setting $W_{A(\mathcal{A})} := H_n \otimes V / \ker A(\mathcal{A})$, we define a skew symmetric isomorphism $q_{A(\mathcal{A})} : W_A \xrightarrow{\simeq} W_A^\vee$ as in (9) above, and also the morphism of sheaves $a_{A(\mathcal{A})} : H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to W_{A(\mathcal{A})} \otimes \mathcal{O}_{\mathbb{P}^3}$ defined by (10) where we set $A = A(\mathcal{A})$. This yields a sequence of morphisms of sheaves as in (11)

$$\mathcal{M}_{A(\mathcal{A})}: \quad 0 \to H_n \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a_{A(\mathcal{A})}} W_A \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{a^\vee_{A(\mathcal{A})} \circ q_{A(\mathcal{A})}} H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0, \tag{43}$$

Note that the condition (42) on $A(\mathcal{A})$ implies that the sequence (43) is exact in the middle term. Indeed, the composition $\mathbf{a} := a^\vee_{A(\mathcal{A})} \circ q_{A(\mathcal{A})} \circ a_{A(\mathcal{A})}$ is an antiselfdual morphism, i.e. $\mathbf{a}^\vee = -\mathbf{a}$, since $q_{A(\mathcal{A})}$ is antiselfdual. This implies that this point $\mathbf{a}$ considered as a vector in the space $\mathrm{Hom}(H_n \otimes \mathcal{O}_{\mathbb{P}^3}(-1), H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1)) \simeq H_n^\vee \otimes H_n^\vee \otimes S^2V^\vee$, lies in fact in its subspace $\wedge^2 H_n^\vee \otimes S^2 V^\vee$. On the other hand, from the definition of $A(\mathcal{A})$ and $a_{A(\mathcal{A})}$ it follows directly that $\mathbf{a}$ is just the

image of the point $A(\mathcal{A}) \in \wedge^2(H_n^\vee \otimes V^\vee)$ under the projection onto the second direct summand of the decomposition (8). Hence (42) implies $\mathbf{a} = 0$.

Thus (43) is a complex, and we obtain the cohomology sheaf of its middle term

$$E(\mathcal{A}) := \ker(a^\vee_{A(\mathcal{A})} \circ q_{A(\mathcal{A})}) / \operatorname{Im} a_{A(\mathcal{A})}. \tag{44}$$

Next, note that the condition $(iv)_\Gamma$ immediately implies that the matrix $\widetilde{\mathcal{A}}$ defined in (29) has rank $2n+2$, hence by (31) and (30) it follows that $\mathrm{rk}A(\mathcal{A}) \le 2n+2$. On the other hand, multiplying $A(\mathcal{A})$ by a nondegenerate matrix

$$D = \begin{pmatrix} \mathbf{1} & \mathbf{0} & & \\ -\mathbf{0} & \mathbf{1} & & \\ B_1 & -A_1 & \mathbf{1} & \mathbf{0} \\ B_2 & -A_2 & \mathbf{0} & \mathbf{1} \end{pmatrix}$$

and using the relations (36) and (33), we obtain

$$D \cdot A(\mathcal{A}) = \begin{pmatrix} \mathbf{0} & \mathbf{1} & A_1 & A_2 \\ -\mathbf{1} & \mathbf{0} & B_1 & B_2 \\ & & a_1 \wedge b_1 & a_1 b_2^T - b_1 a_2^T \\ & & a_2 b_1^T - b_2 a_1^T & a_2 \wedge b_2 \end{pmatrix}.$$

Now using $(iv)_\Gamma$ we obtain $2n+2 \le \mathrm{rk}(D \cdot A(\mathcal{A})) = \mathrm{rk}A(\mathcal{A})$, hence

$$\mathrm{rk}A(\mathcal{A}) = 2n+2. \tag{45}$$

This implies, in particular, that $\dim W_{A(\mathcal{A})} = 2n+2$, so that, by (43) the sheaf $E(\mathcal{A})$ is a rank 2 vector bundle, and from the monad (43) it follows that $h^1(E(\mathcal{A})(-2)) = 0$. This together with $(iii)_\Gamma$ means that $[E(\mathcal{A})] \in I_n$, and we obtain a well-defined map

$$\tau_n : \ \Gamma_n \to I_n : \ \mathcal{A} \mapsto [E(\mathcal{A})]. \tag{46}$$

Consider the projective planes $\mathbb{P}^2_{(1)} := \mathrm{Span}(\mathbf{k}e_1, \mathbf{k}e_2, \mathbf{k}e_3)$ and $\mathbb{P}^2_{(2)} := \mathrm{Span}(\mathbf{k}e_1, \mathbf{k}e_2, \mathbf{k}e_4)$ and the projective line $L = \mathbb{P}^2_{(1)} \cap \mathbb{P}^2_{(2)}$ in $\mathbb{P}^3$. It is known from Barth [B1, Theorems 1 and 3], for any $[E] \in I_n$ the restriction of E onto a general projective line $\mathbb{P}^1 \subset \mathbb{P}^3$ is trivial; respectively, restriction of E onto a general projective plane $\mathbb{P}^2 \subset \mathbb{P}^3$ is stable, unless $n = 1$. Thus, for $n \ge 2$,

$$I_n^0 := \{[E] \in I_n \mid E|_{\mathbb{P}^2_{(i)}} \text{ is stable for } i = 1,2, \text{ and } E|_L \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}\} \tag{47}$$

is a dense open subset of I_n. Respectively, for $n \ge 2$,

$$MI_n^0 := \pi^{-1}(I_n^0) \tag{48}$$

is a dense open subset of MI_n.

It follows from [B2, Theorem 1] that the condition $E(\mathcal{A}) \in I_n^0$ is equivalent to the condition $(iv)_\Gamma$ on $\mathcal{A}$. Hence $\tau_n(\Gamma_n) \subset I_n^0$. Moreover, it is shown by Barth in [B3, Section 8] that

$$\tau_n(\Gamma_n) = I_n^0. \tag{49}$$

Hence we have a well-defined map

$$\kappa_n : \ \Gamma_n \to MI_n^0 : \ \mathcal{A} \mapsto A(\mathcal{A}) \tag{50}$$

fitting in the diagram

$$\begin{tikzcd} \Gamma_n \arrow[r, "\kappa_n"] \arrow[dr, "\tau_n"'] & MI_n^0 \arrow[d, "\pi_n"] \\ & I_n^0. \end{tikzcd} \tag{51}$$

The orthogonal group $O_n = O(H_n, \mathbf{C}) := \{g \in GL(H_n) \mid g \cdot g^T = \mathbf{1}_n\}$ and the symplectic group Sp_2 naturally act on $(S^2 H_n^\vee)^{\times 4} \times (H_n^\vee)^{\times 4}$:

$$g\cdot(A_1, A_2, B_1, B_2, a_1, a_2, b_1, b_2) = (g\cdot A_1\cdot g^T, g\cdot A_2\cdot g^T, g\cdot B_1\cdot g^T, g\cdot B_2\cdot g^T, g\cdot a_1, g\cdot a_2, g\cdot b_1, g\cdot b_2) \tag{52}$$

for $g \in O(n)$, and

$$g\cdot(A_1, A_2, B_1, B_2, a_1, a_2, b_1, b_2) = (A_1, A_2, B_1, B_2, sa_1 + ub_1, sa_2 + ub_2, ta_1 + vb_1, ta_2 + vb_2), \tag{53}$$

for $g = \begin{pmatrix} s & t \\ u & v \end{pmatrix} \in Sp_2$. These two actions commute, hence we have a $O_n \times Sp_2$-action on $(S^2 H_n^\vee)^{\times 4} \times (H_n^\vee)^{\times 4}$. The subgroup $\{(1,1), (-1,-1)\} \simeq \mathbb{Z}_2$ acts trivially, so we get an action of

$$H := O_n \times Sp_2/\{(1,1), (-1,-1)\}$$

on $(S^2 H_n^\vee)^{\times 4} \times (H_n^\vee)^{\times 4}$. Γ_n is clearly H-invariant, i.e. H acts naturally on Γ_n.

Now consider the vector space W_{2n+2} with the symplectic structure $q : W_{2n+2} \xrightarrow{\cong} W_{2n+2}^\vee$ used in the definition of the space MI_n^{sp}. Choose a basis of W_{2n+2} in which q is represented by the matrix Q from (30). Now, under the above choice of the bases in the vector spaces H_n, V and W_{2n+2}, the $(2n+2) \times 4n$-matrix $\widetilde{\mathcal{A}}$ introduced in (29) defines a homomorphism

$$\gamma(\mathcal{A}) : \ H_n \otimes V \to W_{2n+2}. \tag{54}$$

Namely, the homomorphisms $H_n \otimes \mathbf{k}e_i \to W_{2n+2}, \ i = 1,2,3,4$, are given by the columns of $\widetilde{\mathcal{A}}$. Conversely, the point $\mathcal{A}$ is uniquely recovered from $\widetilde{\mathcal{A}}$, i.e. from the point $\gamma(\mathcal{A})$. Hence, we obtain an embedding

$$j_n : \Gamma_n \hookrightarrow U = H_n^\vee \otimes W_{2n+2} \otimes V^\vee.$$

Moreover, setting

$${}^0MI_n^{sp} := \rho_n^{-1}(MI_n^0)$$

and comparing the definitions of ${}^0MI_n^{sp}$ and Γ_n, we obtain that

$$j_n(\Gamma_n) \subset {}^0MI_n^{sp}.$$

By construction, j_n fits in the following diagram extending the diagram (51):

$$\begin{tikzcd} \Gamma_n \arrow[r, hook, "j_n"] \arrow[dr, "\kappa_n"] \arrow[ddr, "\tau_n"'] & {}^0MI_n^{sp} \arrow[d, "\rho_n"] \\ & MI_n^0 \arrow[d, "\pi_n"] \\ & I_n^0 \end{tikzcd} \tag{55}$$

Identifying Γ_n with $j_n(\Gamma_n)$ and setting $MO_n := \rho_n(\Gamma_n), \ r_n := \rho_n|_{\Gamma_n}, \ p_n := \pi_n|_{\Gamma_n}$, we obtain the diagram of principal bundles in étale topology similar to the diagram (27)

$$\begin{tikzcd} \Gamma_n \arrow[r, "Sp_2-\text{bundle}", "r_n"'] \arrow[rr, bend right, "G'-\text{bundle}", "\tau_n"'] & MO_n \arrow[r, "O_n/\{\pm 1\}-\text{bundle}", "p_n"'] & I_n^0 \end{tikzcd}. \tag{56}$$

Next, following Barth [B3, Section 8] consider the monomorphism of groups

$$O_n \times Sp_2 \hookrightarrow GL(H_n) \times Sp(W_{2n+2}, q) : \ (g, \begin{pmatrix} s & t \\ u & v \end{pmatrix}) \mapsto (g, \begin{pmatrix} g & & & \\ & g & & \\ & & s & u \\ & & t & v \end{pmatrix}). \tag{57}$$

This monomorphism commutes with multiplication by (-1,1), hence it descends to a monomorphism

$$H \hookrightarrow G. \tag{58}$$

such that the diagram (27) restricted onto ${}^0MI_n^{sp}$ and the diagram (56) join into a H-equivariant diagram of principal bundles in étale topology

$$\begin{tikzcd}[column sep=huge]
{}^0MI_n^{sp} \arrow[r, "\rho_n", "Sp(W_{2n+2},q)-\text{bundle}"'] \arrow[rr, bend left, "G-\text{bundle}", "\chi_n"'] & MI_n^0 \arrow[r, "\pi_n", "GL(H_n)/\{\pm1\}-\text{bundle}"'] & I_n^0 \arrow[d, equal] \\
\Gamma_n \arrow[u, hook, "j_n"] \arrow[r, "Sp_2-\text{bundle}", "r_n"'] \arrow[rr, bend right, "H-\text{bundle}", "\tau_n"'] & MO_n \arrow[u, hook, "i_n"'] \arrow[r, "O_n/\{\pm1\}-\text{bundle}", "p_n"'] & I_n^0
\end{tikzcd} \tag{59}$$

6. Affine bundle structure

Fix a 3-dimensional subspace of V,

$$V_3 := \mathrm{Span}(\mathbf{k}e_2, \mathbf{k}e_3, \mathbf{k}e_4), \tag{60}$$

and the corresponding projective plane in $\mathbb{P}(V)$:

$$\mathbb{P}^2 = P(V_3).$$

Let X_n be the moduli space of stable rank-2 vector bundles with $c_1 = 0$ and $c_2 = n$ on $\mathbb{P}^2$ for $n \geq 2$. According to Barth [B1, Theorem 3], the restriction of a stable rank-2 vector bundle over $\mathbb{P}^3$ onto a general projective plane in $\mathbb{P}^3$ is also stable. Hence the following is a well defined rational map:

$$\phi_n : \ I_n \dashrightarrow X_n : \ [E] \mapsto [E|_{\mathbb{P}^2}].$$

Set

$$I_n^* := \{[E] \in I_n^0 \mid \phi_n \text{ is regular at } [E]\}$$

Note that the morphism $\phi_n : I_n^* \to X_n$ is dominating. In fact, under the standard identifications $T_{[E]}I_n^* = H^1(E \otimes E)$ and $T_{[E|_{\mathbb{P}^2}]}X_n = H^1(E \otimes E|_{\mathbb{P}^2})$, the differential $d\phi_n|_{[E]}$ of ϕ_n at an arbitrary point $[E] \in I_n^*$ coincides with the restriction map res in the cohomology long exact sequence

$$... \to H^1(E \otimes E) \overset{res}{\to} H^1(E \otimes E|_{\mathbb{P}^2}) \to H^2(E \otimes E(-1)) \to ... \tag{61}$$

of the exact triple $0 \to E \otimes E(-1) \to E \otimes E \to E \otimes E|_{\mathbb{P}^2} \to 0$. Now take $[E] \in I_n^*$ to be a t'Hooft bundle, i.e. a bundle defined as an extension $0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \to E \to \mathcal{I}_{Z,\mathbb{P}^3}(1) \to 0$, where Z is a disjoint union of $n+1$ lines in $\mathbb{P}^3$ considered as a reduced subscheme of $\mathbb{P}^3$. Besides, chose the basis $e_1, ..., e_4$ of V in such a way that $E|_{\mathbb{P}^2}$ is stable. It is well known and can be easily computed (see, e.g., [NT]) that $[E]$ is a smooth point of I_n^* and that $H^2(E \otimes E(-1)) = 0$. Since X_n is a smooth variety of dimension $4n-3$ and $\dim I_n^* = 8n-3$, it follows from (61) that $d\phi_n$ is surjective at the point $[E]$, hence ϕ_n is dominating. Moreover, ϕ_n is a smooth morphism at the point $[E]$ and the fibre $\phi_n^{-1}([E|_{\mathbb{P}^2}])$ is smooth of dimension

$$\dim_{[E]} \phi_n^{-1}([E|_{\mathbb{P}^2}]) = 4n \tag{62}$$

near $[E]$.

Set $X_n^* := \phi_n(I_n^*)$, so that we have a surjection

$$\phi_n : \ I_n^* \twoheadrightarrow X_n^*. \tag{63}$$

Now, for any point $B \in S^2H_n^\vee \otimes \wedge^2 V_3^\vee$ considered as a homomorphism $B : H \otimes V_3 \to H^\vee \otimes V_3^\vee$, define the vector space W_B with the symplectic structure $q_B : W_B \overset{\simeq}{\to} W_B^\vee$ and the morphism of $\mathcal{O}_{\mathbb{P}^2}$-sheaves $a_B : H_n \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \to W_B \otimes \mathcal{O}_{\mathbb{P}^2}$ in the same way as we did for W_A, q_A and a_A (we set $W_B := (\operatorname{Im} B)^\vee$, etc.; see (9), (10)).

Similarly to (13), set

(64)
$$MX_n := \left\{ B \in S^2H_n^\vee \otimes \wedge^2 V_3^\vee \;\middle|\; \begin{array}{l} (i)\ \ \mathrm{rk}(B : H_n \otimes V_3 \to H_n^\vee \otimes V_3^\vee) = 2n+2, \\ (ii)\ \text{the morphism } a_B^\vee : W_B^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \to H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \\ \qquad \text{defined by } B \text{ in (10) is surjective,} \\ (iii)\ h^0(E_2(B)) = 0,\ \text{where } E_2(B) := \ker(a_B^\vee \circ q_B)/\operatorname{Im} a_B \end{array} \right\}$$

The group $GL(H_n)/\{\pm 1\}$ acts freely on MX_n and it is shown by Barth [B2] that $\pi_n' : MX_n \to X_n :\ B \mapsto [E(B)]$ is a principal $GL(H_n)/\{\pm id\}$-bundle in the étale topology.

Note that, under the choice of the basis of H_n as before, respectively, of the basis $e_1, ..., e_4$ of V satisfying (60) we can represent any point $B \in MX_n$ by the 3×3 block matrix, consisting of blocks of size $n \times n$,

$$B = \begin{pmatrix} \mathbf{0} & B_1 & B_2 \\ -B_1 & \mathbf{0} & C \\ -B_2 & -C & \mathbf{0} \end{pmatrix} \,, \tag{65}$$

which is obtained as the right lower corner of $A(\mathcal{A})$ in (40). Set $MX_n := \theta_n^{-1}(X_n^*)$, respectively, $MI_n^* := \pi_n^{-1}(I_n^*)$. We then obtain a well defined morphism

$$\Phi_n :\ MI_n^* \to X_n^* \,, \quad \begin{pmatrix} \mathbf{0} & A_0 & A_1 & A_2 \\ -A_0 & \mathbf{0} & B_1 & B_2 \\ -A_1 & -B_1 & \mathbf{0} & C \\ -A_2 & -B_2 & -C & \mathbf{0} \end{pmatrix} \mapsto \begin{pmatrix} \mathbf{0} & B_1 & B_2 \\ -B_1 & \mathbf{0} & C \\ -B_2 & -C & \mathbf{0} \end{pmatrix} . \tag{66}$$

It fits in the commutative diagram of pricipal $GL(H_n)/\{\pm id\}$-bundles

$$\begin{array}{ccc} MI_n^* & \overset{\Phi_n}{\twoheadrightarrow} & MX_n^* \\ {\scriptstyle \pi_n}\downarrow & & \downarrow{\scriptstyle \pi_n'} \\ I_n^* & \overset{\phi_n}{\twoheadrightarrow} & X_n^*. \end{array} \tag{67}$$

Note that the morphism Φ is surjective since ϕ_n is surjective.

Now repeat the constructions of Section 4 for the vector space

$$U' := H_n^\vee \otimes W_{2n+2} \otimes V_3^\vee$$

Namely, consider the open subset $U_0' := \{\delta \in U' |\ \delta$ understood as a homomorphism $H_n \otimes V \to W_{2n+2}$ is surjective$\}$ and, for an arbitrary point $\delta \in U_0'$ consider the composition

$$B(\delta) : H_n \otimes V_3 \overset{\delta}{\twoheadrightarrow} W_{2n+2} \underset{\sim}{\overset{q}{\to}} W_{2n+2}^\vee \overset{\delta^\vee}{\hookrightarrow} H_n^\vee \otimes V_3^\vee, \quad A(\delta) \in \wedge^2(H_n^\vee \otimes V_3^\vee). \tag{68}$$

Similarly to (17), the condition $\delta \in U_0'$ is equivalent to the condition that

$$\mathrm{rk} B(\delta) = \dim W_{2n+2} = 2n+2.$$

Besides, consider the composition

$$\beta(\delta) : H_n \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \overset{u}{\to} H_n \otimes V \otimes \mathcal{O}_{\mathbb{P}^2} \overset{\delta}{\to} W_{2n+2} \otimes \mathcal{O}_{\mathbb{P}^2}. \tag{69}$$

Define the space
(70)

$$MX_n^{sp} := \left\{ \delta \in U' \;\middle|\; \begin{array}{l} (i)\ \mathrm{rk} B(\delta) = 2n+2, \\ (ii)\ \beta(\delta)^\vee \circ q \circ \beta(\delta) = 0, \\ (iii)\ \text{the morphism } \beta(\delta)^\vee : W_{2n+2}^\vee \otimes \mathcal{O}_{\mathbb{P}^2} \to H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^2}(1) \text{ is surjective,} \\ (iv)\ h^0(E(\delta)) = 0, \text{ where } E(\delta) := \ker(\beta(\delta)_B^\vee \circ q)/\operatorname{Im}\beta(\delta) \end{array} \right\}$$

Note that the conditions (ii)-(iv) in the definition of MX_n^{sp} mean that, for any $\delta \in MX_n^{sp}$, the sequence

$$0 \to H_n \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \overset{\beta(\delta)}{\to} W_{2n+2} \otimes \mathcal{O}_{\mathbb{P}^2} \underset{\sim}{\overset{q}{\to}} W_{2n+2}^\vee \otimes \mathcal{O}_{\mathbb{P}^2} \overset{\beta(\delta)^\vee}{\to} H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^2}(1) \to 0 \tag{71}$$

is a monad (of the same form as (11)) with the cohomology sheaf

$$E = E(\delta) = \ker(\beta(\delta)_A^\vee \circ q)/\operatorname{Im}\beta(\delta), \quad [E] \in X_n. \tag{72}$$

Next, the group $\widetilde{G} := GL(H_n) \times Sp(W_{2n+2}, q)$ naturally acts on the vector space U, hence on MX_n^{sp}, and it is well known (see, e.g., [BH, p. 341, Remark 2]) that the stabilizer in $\widetilde{G}$ of any point $\delta \in MX_n^{sp}$ coincides with the subgroup $\{(id_{H_n}, id_{W_{2n+2}}), (-id_{H_n}, -id_{W_{2n+2}})\} \simeq \mathbb{Z}_2$. Hence the group $G := \widetilde{G}/\mathbb{Z}_2$ acts freely on MX_n^{sp}, and in accordance with (72) we obtain the natural isomorphism $MX_n^{sp}/G \overset{\sim}{\to} X_n$ such that the projection

$$\rho_n' : \ MX_n^{sp} \to X_n : \ \delta \mapsto [E(\delta)] \tag{73}$$

is a principal G-bundle in the étale topology.

Let $pr_2 : \wedge^2(H_n^\vee \otimes V_3^\vee) \to \wedge^2 H_n^\vee \otimes S^2 V_3^\vee$ be the projection onto the second direct summand of the decomposition (8). Comparing (68) and (69) we see that the condition (ii) in (71) can be rewritten as

$$pr_2(B(\delta)) = 0, \tag{74}$$

i.e., equivalently, as

$$B(\delta) \in S^2 H_n^\vee \otimes \wedge^2 V_3^\vee. \tag{75}$$

Whence, comparing the conditions (i)-(iv) in (71) with the conditions (i)-(iii) in (64) we obtain a well-defined morphism

$$\rho_n' : \ MX_n^{sp} \to MX_n : \ \delta \mapsto B(\delta). \tag{76}$$

Note that the natural monomorphism of groups $Sp(W_{2n+2}, q) \hookrightarrow \widetilde{G}$ gives an $Sp(W_{2n+2}, q)$-action on MX_n^{sp} and map ρ_n' is clearly $Sp(W_{2n+2}, q)$-invariant. Show that, in fact, $MX_n \simeq MX_n^{sp}/Sp(W_{2n+2}, q)$ and $\rho_n' : \ MX_n^{sp} \to MX_n$ is a principal $Sp(W_{2n+2}, q)$-bundle in the étale topology. Indeed, for a given point $B \in MX_n$ pick an isomorphism $\psi : W_{2n+2} \overset{\simeq}{\to} W_B$ making commutative the diagram

$$\begin{tikzcd} W_{2n+2} \arrow[r, "q", "\simeq"'] \arrow[d, "\psi"', "\simeq"] & W_{2n+2}^\vee \\ W_A \arrow[r, "q_B", "\simeq"'] & W_B^\vee \arrow[u, "\psi^\vee", "\simeq"'] \end{tikzcd} \tag{77}$$

Now the isomorphism ψ is defined uniquely up to a $Sp(W_{2n+2}, q)$-action on W_{2n+2}. This means that the monad (11) defined by the point B lifts to a monad (71), i.e. to a point $\delta \in {\rho'}_n^{-1}(B)$ uniquely up to a $Sp(W_{2n+2}, q)$-action. Hence ${\rho'}_n^{-1}(B)$ is a principal homogeneous $Sp(W_{2n+2}, q)$-space, and we get the above statement. Thus, denoting by χ_n' the map $MX_n^{sp} \to X_n : \delta \mapsto$

$[E(\delta)]$, we obtain the diagram of principal bundles in the étale topology

(78)
$$\begin{tikzcd} MX_n^{sp} \arrow[r, "Sp(W_{2n+2},q)-\text{bundle}", "\rho'_n"'] \arrow[rr, bend right, "G-\text{bundle}", "\chi'_n"'] & MX_n \arrow[r, "GL(H_n)/\{\pm1\}-\text{bundle}", "\pi'_n"'] & X_n . \end{tikzcd}$$

Now consider the sets

$$^*MI_n^{sp} := \rho_n^{-1}(MI_n^*), \quad ^*MX_n^{sp} := {\rho'}_n^{-1}(MX_n^*).$$

It follows from the above that the projection $U \to U'$ induced by the injection $V_3 \hookrightarrow V$, when restricted to $^*MI_n^{sp}$, gives a well defined morphism

$$\Psi_n : \ ^*MI_n^{sp} \to {}^*MX_n^{sp}.$$

Hence we have the following diagram of principal bundles in the étale topology

(79)
$$\begin{tikzcd} ^*MI_n^{sp} \arrow[r, two heads, "\Psi_n"] \arrow[d, "\rho_n"'] & ^*MX_n^{sp} \arrow[d, "\rho'_n"'] \\ ^*MI_n \arrow[r, two heads, "\Phi_n"] \arrow[d, "\pi_n"'] & ^*MX_n \arrow[d, "\pi'_n"'] \\ I_n^* \arrow[r, two heads, "\phi_n"] & X_n^*. \end{tikzcd}$$

Note that the morphism Ψ_n is surjective since Φ_n is surjective.

Now consider the $O_n/\{\pm1\}$-bundle $p_n : MO_n \to I_n^0$ and the H-bundle from diagram (59) and set

$$MO_n^* := p_n^{-1}(I_n^*), \quad \Gamma_n^* := r_n^{-1}(MO_n^*)$$

Then the diagram (79) extends to a commutative diagram of principal bundles in the étale topology

(80)
$$\begin{tikzcd} \Gamma_n^* \arrow[r, hook] \arrow[d, "r_n"'] & ^*MI_n^{sp} \arrow[r, two heads, "\Psi_n"] \arrow[d, "\rho_n"'] & ^*MX_n^{sp} \arrow[d, "\rho'_n"'] \\ MO_n^* \arrow[r, hook] \arrow[d, "p_n"'] & ^*MI_n \arrow[r, two heads, "\Phi_n"] \arrow[d, "\pi_n"'] & ^*MX_n \arrow[d, "\pi'_n"'] \\ I_n^* \arrow[r, equal] & I_n^* \arrow[r, two heads, "\phi_n"] & X_n^*. \end{tikzcd}$$

Now set

$$\Sigma_n := \Psi_n(\Gamma_n^*), \quad MXO_n := \Phi_n(MO_n^*)$$

and let $r'_n : \Sigma_n \to MXO_n$, $p'_n : MXO_n \to X_n^*$ and $\tau'_n := p'_n \circ r'_n$ be the induced projections. From (80) follows the diagram of principal bundles in the étale topology

(81)
$$\begin{tikzcd} \Sigma_n \arrow[r, "r'_n", "Sp_2-\text{bundle}"'] \arrow[rr, bend left, "H-\text{bundle}", "\tau'_n"'] & MXO_n \arrow[r, "\pi'_n", "O_n)/\{\pm1\}-\text{bundle}"'] & X_n^* . \\ \Gamma_n^* \arrow[u, "\Psi_n"] \arrow[r, "Sp_2-\text{bundle}", "r_n"'] \arrow[rr, bend right, "H-\text{bundle}", "\tau_n"'] & MO_n^* \arrow[u, "\Phi_n"] \arrow[r, "O_n/\{\pm1\}-\text{bundle}", "p_n"'] & I_n^* \arrow[u, "\phi_n"'] \end{tikzcd}$$

Here similarly to the embedding $\Gamma_n^* \hookrightarrow (S^2H_n^\vee)^{\times 4} \times (H_n^\vee)^{\times 4}$ (see (34)) we have a natural H-equivariant embedding

$$\Sigma_n \hookrightarrow (S^2H_n^\vee)^{\times 3} \times (H_n^\vee)^{\times 4} \tag{82}$$

such that the following diagram is commutative

$$\begin{tikzcd} \Gamma_n^* \arrow[r, hook] \arrow[d, "\Psi_n"] & (S^2H_n^\vee)^{\times 4} \times (H_n^\vee)^{\times 4} \arrow[d, "\widetilde{\Psi}_n"] \\ \Sigma_n \arrow[r, hook] \arrow[d, "r_n'"] & (S^2H_n^\vee)^{\times 3} \times (H_n^\vee)^{\times 4} \arrow[d, "pr"] \\ MXO_n \arrow[r, hook] & S^2H_n^\vee \otimes \wedge^2 V^\vee, \end{tikzcd} \tag{83}$$

where $\widetilde{\Psi}_n$ is the the natural map obtained using (33)

$$\widetilde{\Psi}: \ (A_1, A_2, B_1, B_2, a_1, a_2, b_1, b_2) \mapsto (B_1, B_2, A_1B_2 - B_1A_2 + a_1b_2^T - b_1a_2^T, a_1, a_2, b_1, b_2) \tag{84}$$

and pr is the composition of the projection onto the first factor $pr_1: \ (S^2H_n^\vee)^{\times 3} \times (H_n^\vee)^{\times 4} \to (S^2H_n^\vee)^{\times 3}$ and the isomorphism

$$(S^2H_n^\vee)^{\times 3} \xrightarrow{\simeq} S^2H_n^\vee \otimes \wedge^2 V^\vee \ , \ \ (B_1, B_2, C) \mapsto \begin{pmatrix} \mathbf{0} & B_1 & B_2 \\ -B_1 & \mathbf{0} & C \\ -B_2 & -C & \mathbf{0} \end{pmatrix}$$

(see (65)). From (83), (84) and the equations (36)-(38) of Γ_n in $(S^2H_n^\vee)^{\times 4} \times (H_n^\vee)^{\times 4}$ we obtain the following description of the fibre of the surjection $\Psi_n : \Gamma_n^* \to \Sigma_n$:

(85)

$$\Psi_n^{-1}(B_1, B_2, C, a_1, a_2, b_1, b_2) \simeq \left\{ (A_1, A_2) \in (S^2H_n^\vee)^{\times 2} \ \middle| \ \begin{array}{l} [A_1, B_1] + a_1 \wedge b_1 = 0, \\ [A_2, B_2] + a_2 \wedge b_2 = 0, \\ A_1B_2 - B_1A_2 + a_1b_2^T - b_1a_2^T = C \\ \text{plus some open conditions.} \end{array} \right\}$$

Proof of Theorem 1.1. Since the equations (85) on A_1, A_2 are linear, it follows that the fibre of the morphism $\Psi_n : \Gamma_n^* \to \Sigma_n$ is an open subset of an affine space $\mathbb{A}^r$. Note that by the H-equivariant diagram (81) of principal bundles in the étale topology this morphism Ψ_n is a H-equivariant and surjective. On the other hand, both X_n^* and I_n^* are irreducible of dimensions $4n-3$ and $8n-3$ respectively, hence it follows that the minimal value of the dimension r of the fibre of Ψ_n is $4n$. Moreover, there exists a dense open H-equivariant subset of Σ_n,

$$\Sigma_n' := \{B \in \Sigma_n \mid \Psi_n^{-1}(B) \overset{\text{open}}{\hookrightarrow} \mathbb{A}^{4n}\}.$$

Set $\Gamma_n' := \Psi_n^{-1}(\Sigma_n')$. Over Σ_n, the matrices A_1, A_2 form a vector bundle of rank $2n^2$ whose structure group is reduced to GL_n, and this action restricts to the affine spaces $\mathbb{A}^{4n}$, cut out by the closed conditions of formula (85), as an action by affine transformations. Also, the elements of H, restricted to these affine spaces, act by affine isomorphisms. Thus,

$$\Psi_n : \Gamma_n' \to \Sigma_n'$$

is an open piece in an H-equivariant affine bundle. Setting $X_n' := \tau_n'(\Sigma_n'), \quad I_n' := \phi^{-1}(X_n')$, we have a H-equivariant diagram of principal H-bundles in the étale topology

$$\begin{tikzcd} \Gamma_n' \arrow[r, two heads, "\Psi_n"] \arrow[d, "\tau_n"'] & \Sigma_n' \arrow[d, "\tau_n'"] \\ I_n' \arrow[r, two heads, "\phi_n"] & X_n'. \end{tikzcd} \tag{86}$$

Here H acts freely on Γ'_n, respectively, on Σ'_n. By the Noname Lemma, upon shrinking X'_n, the H-equivariant affine bundle descends as the direct product $\mathbb{A}^{4n}_{\mathbf{k}} \times X'_n$, and I'_n is an open set in it. Thus I'_n is birational to $\mathbb{A}^{4n}_{\mathbf{k}} \times X'_n$. As X'_n is an open subset of X_n, and X_n is rational by [M], hence I'_n is also rational. Since I'_n is an open subset of I_n, it follows that I_n is rational as well.

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